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A WEDDERBURN THEOREM FOR ALTERNATIVE ALGEBRAS WITH IDENTITY OVER COMMUTATIVE RINGS

BY

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ABSTRACT. In this paper, we study alternative algebras Λ over a commutative, associative ring R with identity. When Λ is finitely generated as an R-module, we define the radical J of Λ. We show that matrix units and split Cayley algebras can be lifted from Λ/J to Λ when R is a Hensel ring. We also prove the following Wedderburn theorem: *Let Λ be an alternative algebra over a complete local ring R of equal characteristic. Suppose Λ is finitely generated as an R-module, and Λ/J is separable over $\bar{R}$ ($\bar{R}$ the residue class field of R). Then there exists an $\bar{R}$-subalgebra S of Λ such that $S + J = \Lambda$ and $S \cap J = 0$.*

Introduction. Let Λ denote an alternative algebra over a field R. Let J denote the radical of Λ. It is well known that the Wedderburn theorem holds for alternative algebras. That is, if Λ is finite dimensional over R, and Λ/J is separable over R, then there exists a separable subalgebra S of Λ such that $S \oplus J = \Lambda$. If Λ is associative, the author in [4] has generalized this result to the case where R is a split Hensel ring. The purpose of this paper is to obtain similar results for alternative algebras over complete local rings R of equal characteristic. The precise result is as follows: Let R be a complete local ring of equal characteristic. Let $\bar{R}$ denote the residue class field of R. Let Λ be an alternative algebra over R such that Λ is finitely generated as an R-module and, Λ/J (J the radical of Λ) is separable over R. Then there exists an $\bar{R}$-subalgebra S of Λ such that $S + J = \Lambda$ and $S \cap J = 0$.

In order to prove this result, we must carefully define what we mean by the radical J of Λ. Once this has been done, the results follow much as in the associative case.

Preliminaries. Throughout the rest of this paper, R will denote an associative, commutative ring with identity 1. Λ will always denote an alternative ring with identity. Thus, Λ is a not necessarily associative or commutative ring which satisfies the following two identities:

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$$x^2y = x(xy), \qquad yx^2 = (yx)x \quad \text{for all } x, y \in \Lambda. \tag{1}$$

It is not absolutely necessary to assume Λ has an identity under multiplication. In the process of proving the main results of this paper, we could always first adjoin an identity to Λ. But for convenience we shall always assume Λ contains an identity.

By the center $C(\Lambda)$ of Λ, we shall, as usual, mean the collection of all elements c in Λ which both commute and associate with all elements of Λ. We are now ready for the following definition: Λ is an R-algebra if there exists a ring homomorphism $\theta: R \to C(\Lambda)$ such that $\theta(1)$ is the identity of Λ. From now on we shall suppress θ and simply write $r\lambda$ instead of $\theta(r)\lambda$, $r \in R$ and $\lambda \in \Lambda$. If Λ is an R-algebra, then Λ is naturally an R-module. In particular, an ideal $J \subset \Lambda$ is an R-submodule of Λ, i.e. $RJ \subset J$. We say Λ is free, finitely generated, flat etc. over R if Λ is free, finitely generated, flat etc. as an R-module. Throughout this paper, we assume that all ring homomorphisms of any rings in question which have identities take the identity to the identity. By an R-algebra homomorphism from Λ to another alternative algebra Λ' over R, we shall mean an algebra homomorphism which is also an R-module homomorphism.

Many of the results which hold for alternative algebras over fields pass over to similar results in our setting. In particular, the Moufang identities [9, p. 28], Artin's theorem [9, Theorem 3.1] and the Peirce decomposition [9, pp. 32–37] hold for any alternative algebra Λ over R. We shall use these results freely whenever needed.

Now let R be a local ring, i.e. R is a Noetherian ring with exactly one maximal ideal m. Let π_0 denote the natural projection of R onto $\bar{R} = R/m$. We shall say that R is split if there exists a ring homomorphism $\epsilon_0: \bar{R} \to R$ such that $\pi_0\epsilon_0$ is the identity map on $\bar{R}$. It follows from [6, Theorem 9] that any complete local ring of equal characteristic is split. If R is split, then via ϵ_0 we may identify $\bar{R}$ with a subring of R containing 1. If Λ is an alternative algebra over a split local ring R, and J is an ideal in Λ containing $m\Lambda$, then $0 \to J \to \Lambda \to \Lambda/J \to 0$ can naturally be viewed as a short exact sequence of $\bar{R}$-algebras. We shall say that this sequence splits if there exists an $\bar{R}$-algebra homomorphism $\epsilon: \Lambda/J \to \Lambda$ such that $\pi\epsilon$ is the identity map on Λ/J. Here π is of course the natural projection of Λ onto Λ/J.

Finally, we need the definition of a Hensel ring. Suppose R is a local ring (R need not be Noetherian here) with maximal ideal m. Let X be an indeterminate over R and consider the polynomial ring $R[X]$. We have a natural ring homomorphism $\sigma: R[X] \to \bar{R}[X]$ induced by π_0. Namely if $f(X) = \sum r_i X^i \in R[X]$, then $\sigma(f) = \sum \pi_0(r_i) X^i \in \bar{R}[X]$. Let us write $\bar{f}$ for $\sigma(f)$. R is called a Hensel ring if every monic polynomial $f(X) \in R[X]$ satisfies the following condition: If there exist two relatively prime polynomials $g_1(X)$ and $g_2(X) \in \bar{R}[X]$ such that $\bar{f} = g_1 g_2$ and g_1 is monic, then there exist two polynomials $h_1(X)$ and $h_2(X) \in R[X]$ such that $h_1 h_2 = f$, $\bar{h}_1 = g_1$, $\bar{h}_2 = g_2$ and h_1 is monic. It is well known that complete local rings are Hensel rings [8, Theorem 30.4].

The author assumes the reader is familiar with the theory of associative algebras over Hensel rings [3].

I. **The radical of an alternative algebra over a commutative ring.** Let Λ be an alternative algebra over R which is finitely generated. Let $\Omega(R)$ denote the collection of all maximal ideals of R. Then for each $m \in \Omega(R)$, $\Lambda/m\Lambda$ is an alternative algebra over the field R/m. Since Λ is finitely generated as an R-module, $\Lambda/m\Lambda$ is finite dimensional over R/m. Thus, the radical of $\Lambda/m\Lambda$ is well defined and can be taken to be the collection of all properly nilpotent elements in $\Lambda/m\Lambda$ [9, Theorem 3.7]. Let $J(m)$ be the ideal in Λ containing $m\Lambda$ and such that $J(m)/m\Lambda$ is the radical of $\Lambda/m\Lambda$. Thus, $J(m)$ is the full inverse image of the radical of $\Lambda/m\Lambda$ under the natural projection $\Lambda \rightarrow \Lambda/m\Lambda$. We now define the radical J of Λ as follows:

(2) $$J = \bigcap_{m \in \Omega(R)} J(m).$$

We note that if R is a field, then our definition of the radical of Λ agrees with the classical definition for alternative algebras. If we assume Λ is an associative algebra, then J is just the Jacobson radical of Λ [3, Corollary, p. 125]. In any case, we note that we have defined the radical of Λ for only those alternative algebras which are finitely generated over R. We shall need the following facts about J.

Proposition 1. *Let Λ be an alternative algebra which is finitely generated over R. Let J be the radical of Λ. Then if $x \in J$, $1 - x$ is a unit in Λ.*

Proof. If Λ has a generating set of cardinality n over R, then for every $m \in \Omega(R)$ the dimension of $\Lambda/m\Lambda$ over R/m is less than or equal to n. Hence by [9, Theorem 3.7], $J(\Lambda/m\Lambda)^n = 0$. Thus, if $x \in J$, the image $\bar{x}$ of x in $\Lambda/m\Lambda$ is nilpotent, i.e. $\bar{x}^n = 0$ for any m. So,

$$(\bar{1} - \bar{x})(\bar{1} + \bar{x} + \cdots + \bar{x}^{n-1}) = \bar{1} \quad \text{in } \Lambda/m\Lambda.$$

Thus,

(3) $$1 - (1 - x)(1 + x + \cdots + x^{n-1}) \in \bigcap_{m \in \Omega(R)} m\Lambda.$$

So, there is a $z \in \bigcap m\Lambda$ such that $1 = (1 - x)(1 + x + \cdots + x^{n-1}) + z$. Now if $\lambda \in \Lambda$, then $\lambda = [(1 - x)(1 + x + \cdots + x^{n-1})]\lambda + z\lambda$. By Artin's theorem, $R[\lambda, x]$ is associative. Thus, $[(1 - x)(1 + x + \cdots + x^{n-1})]\lambda = (1 - x)(\lambda + \cdots + x^{n-1}\lambda)$. So as R-modules, we have $\Lambda = (1 - x)\Lambda + m\Lambda$ for every $m \in \Omega(R)$. It now follows from Nakayama's lemma [3, Corollary, p. 124] that $\Lambda = (1-x)\Lambda$. So there exists a $\lambda^1 \in \Lambda$ such that $(1-x)\lambda^1 = 1$. In a similar manner, we show $(1-x)$ has a left inverse. Therefore, $1-x$ has a unique two sided inverse. □

An element $x \in \Lambda$ will be called quasi-regular if there exists an element $y \in \Lambda$ such that $x + y - xy = x + y - yx = 0$. The element y is called the quasi-inverse of x and is easily seen to be unique. It follows from [10, Lemma 2] that x is a quasi-regular if and only if $1 - x$ is a unit in Λ. As in the associative theory, we say an ideal $I \subset \Lambda$ is quasi-regular if every element of I is quasi-regular. It follows from Proposition 1 that J is a quasi-regular ideal. We shall need the following proposition:

Proposition 2 (McCrimmon). *Let Λ be an alternative algebra, finitely generated over R. Let J be the radical of Λ and e an idempotent in Λ. Then $J \cap e\Lambda e = eJe$ is a quasi-regular ideal in the subalgebra $e\Lambda e$.*

Proof. It follows easily from the Moufang identities that $e\Lambda e$ is a subalgebra of Λ. Since J is an ideal in Λ, we have $eJe \subset J \cap e\Lambda e$. Let $x \in J \cap e\Lambda e$. Then $x = e\lambda e$ for some $\lambda \in \Lambda$. Now $ex = e(e\lambda e) = e\lambda e = x$. Similarily $xe = x$. Hence, $x = exe \in eJe$. So, $eJe = J \cap e\Lambda e$.

Suppose $x \in J \cap e\Lambda e$. Then $x \in J$. So, x is a quasi-regular element in Λ. Let y be the quasi-inverse of x and consider $eye \in e\Lambda e$. Using the Moufang identities, we have $x(eye) = ((xe)y)e = (xy)e = (x+y)e = xe + ye = x + ye$. If we now multiply by e, we get $e[x(eye)] = x + eye$. Now x and $eye \in e\Lambda e$. Therefore, $x(eye) \in e\Lambda e$, and $e[x(eye)] = x(eye)$. Thus, eye is a right quasi-inverse for x. A similar argument shows eye is a left quasi-inverse for x. Thus, x has a quasi-inverse eye in $e\Lambda e$. Since x was an arbitrary element of eJe, we have shown eJe is a quasi-regular ideal in $e\Lambda e$. □

If Λ is a finitely generated, associative algebra over R, and x is a quasi-regular element of Λ, then the quasi-inverse of x is of the form $r_1x + r_2x^2 + \cdots + r_nx^n$ for $r_i \in R$. We shall need this same result for alternative algebras.

Proposition 3. *Let Λ be an alternative algebra, finitely generated over R. Let x be a quasi-regular element of Λ with quasi-inverse y. Then $y = r_1x + \cdots + r_nx^n$ for some n and $r_i \in R$.*

Proof. Consider the subalgebra of Λ generated by x and y, i.e. $R[x, y]$. Since y is the quasi-inverse of x, $yx = xy = x + y$. Thus, $R[x, y]$ is a commutative, associative ring. Since Λ is finitely generated over R, it follows from [3, Theorem 8] that both x and y satisfy monic polynomials with coefficients in R. Thus, $R[x, y]$ is finitely generated as an R-module. The result now follows from applying [3, Theorem 9] to $R[x, y]$. □

Finally we need to know how the radical behaves under homomorphic images.

Proposition 4. *Let Λ_1 and Λ_2 be two alternative algebras over R which are finitely generated. Let J_1 and J_2 be the radicals of Λ_1 and Λ_2 respectively. Suppose $\sigma\colon \Lambda_1 \to \Lambda_2$ is an R-algebra epimorphism, then $\sigma(J_1) \subset J_2$.*

Proof. For each $m \in \Omega(R)$, let $J_i(m)$ denote the pull back of the radical of $\Lambda_i/m\Lambda_i$ under the natural projection $\Lambda_i \to \Lambda_i/m\Lambda_i$. The map σ induces an R/m-algebra homomorphism $\sigma_m\colon \Lambda_1/m\Lambda_1 \to \Lambda_2/m\Lambda_2$ which is onto. So, we have the following commutative square of epimorphisms:

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\sigma} & \Lambda_2 \\ \downarrow & & \downarrow \\ \Lambda_1/m\Lambda_1 & \xrightarrow{\sigma_m} & \Lambda_2/m\Lambda_2 \end{array}$$

Since the radical of $\Lambda_1/m\Lambda_1$ consists of all properly nilpotent elements and σ_m is onto, σ_m takes the radical of $\Lambda_1/m\Lambda_1$ into the radical of $\Lambda_2/m\Lambda_2$. Since $J_1(m)$ is mapped onto the radical of $\Lambda_1/m\Lambda_1$, it follows that $\sigma(J_1(m)) \subset J_2(m)$. Since this holds for every $m \in \Omega(R)$, we have $\sigma(\bigcap_{m\in\Omega(R)} J_1(m)) \subset \bigcap_{m\in\Omega(R)} J_2(m)$. Thus $\sigma(J_1) \subset J_2$. □

II. Alternative algebras over Hensel rings. In this section, we shall assume that R is a Hensel ring. This type of assumption is inevitable in trying to prove a Wedderburn type theorem because Hensel rings are the only commutative rings which permit idempotents and matrix units to be lifted from Λ/J to Λ.

Theorem 1. *Let Λ be a finitely generated alternative algebra over a Hensel ring R. Let I be an ideal in Λ. If $\bar{e}_1, \cdots, \bar{e}_n$ are pairwise orthogonal idempotents in Λ/I, then there exist pairwise orthogonal idempotents $e_1, \cdots, e_n$ in Λ such that $\pi(e_i) = \bar{e}_i$. Here π is the natural projection of Λ onto Λ/I.*

Proof. Suppose $\bar{e}$ is an idempotent in Λ/I. Let $c \in \Lambda$ such that $\pi(c) = \bar{e}$. Set $S = R[c]$. Then S is a commutative extension of R which by [3, Theorem 8] is finitely generated as an R-module. $I \cap S$ is an ideal in S, and $S/I \cap S$ contains $\bar{e}$. Thus by [3, Theorem 24], S contains an idempotent e such that $\pi(e) = \bar{e}$. Hence, any idempotent in Λ/I can be lifted to Λ.

Now suppose e is an idempotent in Λ, and $\bar{e}_1$ is an idempotent in Λ/I such that $\bar{e}\bar{e}_1 = \bar{e}_1\bar{e} = 0$ $(\bar{e} = \pi(e))$. Then there exists an idempotent $e_1 \in \Lambda$ such that $e_1 e = e e_1 = 0$, and $\pi(e_1) = \bar{e}_1$. The proof of this is as follows: Let $T(e) = \{x \in \Lambda \mid xe = ex = 0\}$, the set of two sided annihilators of e in Λ. Suppose $x, y \in T(e)$. Then $(xy)e = (x, y, e) + x(ye) = (x, y, e) = -(x, e, y) = 0$. Here $(x, y, z) = (xy)z - x(yz)$ is the associator of three elements. Thus, $T(e)$ is a subalgebra of Λ and clearly consists of all elements of the form $\lambda - e\lambda - \lambda e + e\lambda e$, $\lambda \in \Lambda$. Hence

$\pi(T(e)) = \{\bar{x} \in \Lambda/I \mid \bar{x}\bar{e} = \bar{e}\bar{x} = 0\} = T(\bar{e})$. $\bar{e}_1 \in T(\bar{e})$ by hypothesis. Thus, there exists an element $c \in T(e)$ such that $\pi(c) = \bar{e}_1$. Now consider $R\langle c\rangle$ the subalgebra of Λ consisting of all polynomials in c without constant term. Then $R\langle c\rangle \subset T(e)$. It follows from [3, Theorem 21] that $R\langle c\rangle$ contains an idempotent e_1 such that $\pi(e_1) = \bar{e}_1$. Thus, Λ contains an idempotent e_1 with $\pi(e_1) = \bar{e}_1$ and $e_1 e = 0 = ee_1$.

Now suppose $\bar{e}_1, \cdots, \bar{e}_n$ are pairwise orthogonal idempotents in Λ/I. Lift $\bar{e}_1$ to an idempotent $e_1 \in \Lambda$. By the second paragraph of this proof, lift $\bar{e}_2$ to an idempotent e_2 in Λ orthogonal to e_1. Suppose we have lifted $\bar{e}_1, \cdots, \bar{e}_m$, $1 \leq m < n$, to pairwise orthogonal idempotents $e_1, \cdots, e_m$ in Λ. We can lift $\bar{e}_{m+1}$ to an idempotent e_{m+1} in Λ which is orthogonal to $e = \sum_{i=1}^{m} e_i$. Then for $i = 1, \cdots, m$, we have $e_i e_{m+1} = (ee_i e)e_{m+1} = e[e_i(ee_{m+1})] = 0$. Similarly, $e_{m+1}e_i = 0$. Thus, $e_1, \cdots, e_{m+1}$ are pairwise orthogonal, and the proof is completed by induction. □

For future reference, we note that a slight modification of the proof of Theorem 1 yields the following result: Let Λ be an alternative algebra over R which is not necessarily finitely generated as an R-module. Suppose I is an ideal in Λ for which $I^2 = 0$. Then if $\bar{e}_1, \cdots, \bar{e}_n$ are pairwise orthogonal idempotents in Λ/I, there exist pairwise orthogonal idempotents $e_1, \cdots, e_n \in \Lambda$ such that $\pi(e_i) = \bar{e}_i$. In this case, using the same S and $R\langle c\rangle$ as appears in the proof of Theorem 1, we note $I \cap S$ and $I \cap R\langle c\rangle$ are nilpotent ideals in S and $R\langle c\rangle$ respectively. Thus, we may use the results in [7, Proposition 3.4, p. 42] to lift idempotents from $S/I \cap S$ to S and from $R\langle c\rangle/I \cap R\langle c\rangle$ to $R\langle c\rangle$. Hence, Theorem 1 holds in general if $I^2 = 0$.

For the next proposition, we do not require R to be a Hensel ring.

Proposition 5. *Let Λ be an alternative algebra over R. Let I be a quasi-regular ideal in Λ. Let e and f be two pairwise orthogonal idempotents in Λ whose images in Λ/I we denote by $\bar{e}$ and $\bar{f}$. Suppose there exist elements $\bar{a} \in \bar{e}(\Lambda/I)\bar{f}$ and $\bar{b} \in \bar{f}(\Lambda/I)\bar{e}$ such that $\bar{a}\bar{b} = \bar{e}$ and $\bar{b}\bar{a} = \bar{f}$; then there exist elements $a \in e\Lambda f$ and $b \in f\Lambda e$ such that $ab = e$, $ba = f$, $\pi(a) = \bar{a}$, and $\pi(b) = \bar{b}$.*

Here π as usual denotes the natural projection of Λ onto Λ/I.

Proof. It is well known that $(x, e, f) = 0$ if e and f are pairwise orthogonal idempotents. Thus, $e\Lambda f$ and $f\Lambda e$ are unambiguous. Let $a_1 \in e\Lambda f$ and $b_1 \in f\Lambda e$ such that $\pi(a_1) = \bar{a}$, $\pi(b_1) = \bar{b}$. Then $a_1 = e\lambda_1 f$ and $b_1 = f\lambda_2 e$ for some $\lambda_1, \lambda_2 \in \Lambda$. Thus, $a_1 b_1 = (e\lambda_1 f)(f\lambda_2 e) = e \cdot (\lambda_1 f)(f\lambda_2) \cdot e \in e\Lambda e$. Also, $a_1 b_1 - e \in I$. Thus, $e - a_1 b_1 \in I \cap e\Lambda e$ which by the proof of Proposition 2 is a quasi-regular ideal in $e\Lambda e$. Since $e - a_1 b_1$ is quasi-regular in $e\Lambda e$, $e - (e - a_1 b_1) = a_1 b_1$ is a unit in $e\Lambda e$. Thus, there exists an $x \in e\Lambda e$ such that $(a_1 b_1)x = e = x(a_1 b_1)$. Similarly, there exists a $y \in f\Lambda f$ such that $y(b_1 a_1) = f = (b_1 a_1)y$.

Set $a = a_1$ and $b = b_1 x$. Then $a \in e\Lambda f$. Since $x = e\lambda_3 e$, for some $\lambda_3 \in \Lambda$, to show $b \in f\Lambda e$ we must show $(f\lambda_2 e)(e\lambda_3 e) \in f\Lambda e$. Now $f[(f\lambda_2 e)(e\lambda_3 e)] = -(f, f\lambda_2 e, e\lambda_3 e) + [f(f\lambda_2 e)](e\lambda_3 e)$. By the Moufang identities, $(f, f\lambda_2 e, e\lambda_3 e) = (f^2, f\lambda_2 e, e\lambda_3 e) = (f, f\lambda_2 e, f(e\lambda_3 e) + (e\lambda_3 e)f) = 0$ since $f(e\lambda_3 e) = (fe)\cdot\lambda_3\cdot e = 0 = e\cdot\lambda_3\cdot(ef) = (e\lambda_3 e)f$. Also $f(f\lambda_2 e) = f\lambda_2 e$ by Artin's theorem. Hence, $f[(f\lambda_2 e)(e\lambda_3 e)] = (f\lambda_2 e)(e\lambda_3 e)$. Using the same techniques, we can show $[(f\lambda_2 e)(e\lambda_3 e)]e = (f\lambda_2 e)(e\lambda_3 e)$. Hence $b_1 x = (f\lambda_2 e)(e\lambda_3 e) \in f\Lambda e$.

Now $ab = a_1(b_1 x) = -(a_1, b_1, x) + e$. But $-(a_1, b_1, x) = (e\lambda_1 f, e\lambda_3 e, f\lambda_2 e) = (e\lambda_1 f)(e\lambda_3 e)\cdot(f\lambda_2 e) - (e_1\lambda_1 f)\cdot(e\lambda_3 e)(f\lambda_2 e) = [e\cdot(\lambda_1 f)(e\lambda_3)\cdot e]\cdot(f\lambda_2 e) - (e\lambda_1 f)\cdot[e\cdot(\lambda_3 e)(f\lambda_2)\cdot e] = e\cdot[(\lambda_1 f)(e\lambda_3)\cdot e(f\lambda_2 e)] - [(e\lambda_1 f)e\cdot(\lambda_3 e)(f\lambda_2)]\cdot e = 0$. Thus $ab = e$.

In order to show $ba = f$, we first need $yb_1 = b_1 x$. Now the proof above shows $b_1 a_1 b_1 \in f\Lambda e$. From this fact, one can easily argue that $(y, b_1 a_1 b_1, x) = 0$ and $(y, b_1, a_1) = 0$. These are routine arguments like the one above and will be omitted. We then have $yb_1 = y(b_1 e) = y\cdot[b_1\cdot(a_1 b_1)x]$. But we have shown that $(a_1, b_1, x) = 0$. So, $y(b_1 e) = y\cdot[b_1\cdot(a_1 b_1)x] = y\cdot(b_1\cdot a_1(b_1 x)) = y\cdot(b_1 a_1 b_1)x = y(b_1 a_1 b_1)\cdot x = ((yb_1)a_1\cdot b_1)\cdot x = (y(b_1 a_1)\cdot b_1)\cdot x = (fb_1)x = b_1 x$. Thus, $yb_1 = b_1 x$. It now follows that $ba = (b_1 x)a = (yb_1)a = y(b_1 a) = f$.

To finish the proof, we note that $\pi(a) = \bar{a}$ and $\pi(b) = \bar{b}$. □

We can now prove the main result of this section.

Theorem 2. *Let Λ be an alternative algebra over a Hensel ring R. Assume Λ is finitely generated over R, and I is a quasi-regular ideal in Λ. Suppose $\{\bar{e}_{ij} \mid i, j = 1, \cdots, n\}$ is a system of matrix units in Λ/I, then there exists a system of matrix units $\{e_{ij} \mid i, j = 1, \cdots, n\}$ in Λ such that $\pi(e_{ij}) = \bar{e}_{ij}$.*

Proof. Since $\bar{e}_{11}, \cdots, \bar{e}_{nn}$ are pairwise orthogonal idempotents in Λ/I, there exist, by Theorem 1, pairwise orthogonal idempotents $e_1, \cdots, e_n \in \Lambda$ such that $\pi(e_i) = \bar{e}_{ii}$. Now for each $i = 1, \cdots, n$, we have $\bar{e}_{11}\bar{e}_{1i} = \bar{e}_{1i} = \bar{e}_{1i}\bar{e}_{ii}$, $\bar{e}_{ii}\bar{e}_{i1} = \bar{e}_{i1} = \bar{e}_{i1}\bar{e}_{11}$. Thus, $\bar{e}_{1i} \in \bar{e}_{11}(\Lambda/I)\bar{e}_{ii}$, $\bar{e}_{i1} \in \bar{e}_{ii}(\Lambda/I)\bar{e}_{11}$ and $\bar{e}_{1i}\bar{e}_{i1} = \bar{e}_{11}$, $\bar{e}_{i1}\bar{e}_{1i} = \bar{e}_{ii}$. Applying Proposition 5, we get elements $e_{i1} \in e_i\Lambda e_1$ and $e_{1i} \in e_1\Lambda e_i$ such that $e_{i1}e_{1i} = e_i$, $e_{1i}e_{i1} = e_1$, $\pi(e_{1i}) = \bar{e}_{1i}$ and $\pi(e_{i1}) = \bar{e}_{i1}$. Put $e_{11} = e_1$ and $e_{ij} = e_{i1}e_{1j}$ for $i \neq 1$, $j \neq 1$. We shall show that $\{e_{ij} \mid i, j = 1, \cdots, n\}$ are the required matrix units of Λ.

We first note that $e_{ij} \in e_i\Lambda e_j$ because $e_{ij} = e_{i1}e_{1j} = (e_i\lambda_1 e_1)(e_1\lambda_2 e_j)$ for elements $\lambda_1, \lambda_2 \in \Lambda$ [9, 3.17]. A simple argument involving the Moufang identities shows $e_i[(e_i\lambda_1 e_1)(e_1\lambda_2 e_j)] = [(e_i\lambda_1 e_1)(e_1\lambda_2 e_j)]e_j = e_{ij}$. Thus, $e_{ij} \in e_i\Lambda e_j$. We need to show that $e_{ij}e_{jk} = e_{ik}$ for all $i, j, k = 1, \cdots, n$.

Let us first show $e_{ij}e_{kl} = 0$ if $k \neq j$. A linearized form of the Moufang identity [9, 3.5] is as follows:

(4) $$y \cdot (xa)z + y \cdot (za)x = (yx)a \cdot z + (yz)a \cdot x$$

for all x, y, z and $a \in \Lambda$. If we set $y = e_{ij}$, $x = e_k$, $a = e_{kl}$ and $z = e_l$ and use the fact that $e_{ij} \in e_i \Lambda e_j$, (4) becomes

(5) $$e_{ij}e_{kl} + e_{ij} \cdot (e_l e_{kl})e_k = (e_{ij}e_l)e_{kl} \cdot e_k.$$

If $k = l$, then $e_{ij}e_{kl} = e_{ij}e_k = 0$. Thus, without loss of generality we may assume $k \neq l$. In this case, (5) becomes

(6) $$e_{ij}e_{kl} = [(e_{ij}e_l)e_{kl}]e_k.$$

If $l \neq j$, then (6) implies $e_{ij}e_{kl} = 0$. Thus, we may assume $l = j$. In this case, (6) becomes

(7) $$e_{ij}e_{kj} = (e_{ij}e_{kj})e_k.$$

If $k = i$, (7) implies $e_{ij}e_{ij} = (e_{ij}e_{ij})e_i = e_{ij}(e_{ij}e_i) = 0$. If $i = j$, $e_{ij}e_{kj} = e_i e_{kj} = 0$. Thus, we have reduced the proof to showing that $e_{ij}e_{kj} = 0$ when i, j and k are all distinct.

Since i, j and k are all distinct, we get $e_i(e_{ij}e_{kj}) = (e_{ij}, e_i, e_{kj}) + e_{ij}e_{kj} = e_{ij}e_{kj}$. Thus, $e_{ij}e_{kj} \in e_i\Lambda$. Now $0 = (e_j e_{ij})e_{kj} = (e_j, e_{ij}, e_{kj}) + e_j(e_{ij}e_{kj})$. Since $e_{ij}e_{kj} \in e_i\Lambda$, $e_j(e_{ij}e_{kj}) = 0$. Thus, $0 = (e_j, e_{ij}, e_{kj})$. So, $0 = (e_{ij}, e_j, e_{kj}) = (e_{ij}e_j)e_{kj} - e_{ij}(e_j e_{kj}) = e_{ij}e_{kj}$. Therefore, in all cases $e_{ij}e_{kl} = 0$ if $k \neq j$.

It remains to show that $e_{ij}e_{jk} = e_{ik}$. If $i = j$, we have $e_{ij}e_{jk} = e_i e_{ik} = e_{ik}$. If $k = j$, we have $e_{ij}e_{jk} = e_{ij}e_j = e_{ij} = e_{ik}$. Thus, we may assume $i \neq j$ and $j \neq k$. Now $e_{ij}e_{jk} = e_{ij}(e_{j1}e_{1k}) = -(e_{ij}, e_{j1}, e_{1k}) + (e_{ij}e_{j1})e_{1k}$. Using the previous result, one easily argues that $(e_{ij}, e_{j1}, e_{1k}) = 0$. So $e_{ij}e_{jk} = (e_{ij}e_{j1})e_{1k}$.

Now $e_{ij}e_{j1} = (e_{i1}e_{1j})e_{j1} = (e_{i1}, e_{1j}, e_{j1}) + e_{i1}(e_{1j}e_{j1})$. Again one may argue that $(e_{i1}, e_{1j}, e_{j1}) = 0$. We thus get $e_{ij}e_{j1} = e_{i1}$. Thus, $e_{ij}e_{jk} = (e_{ij}e_{j1})e_{1k} = e_{i1}e_{1k} = e_{ik}$. Since $e_{ii} = e_i$ and $\pi(e_{ij}) = \bar{e}_{ij}$, the proof is complete. □

The proof of Theorem 2 requires Theorem 1, Proposition 5 and well known identities for alternative rings. The proof of Proposition 5 does not require Λ to be finitely generated over R. Any ideal I in Λ whose square is zero is clearly a quasi-regular ideal. Thus, using the remark following Theorem 1, we could prove the following analogue of Theorem 2: Let Λ be an alternative algebra over R which is not necessarily finitely generated. Let I be an ideal in Λ such that $I^2 = 0$. Then if $\{\bar{e}_{ij} \mid i, j = 1, \cdots, n\}$ is a system of matrix units in Λ/I, there exists a system of matrix units $\{e_{ij}\} \in \Lambda$ such that $\pi(e_{ij}) = \bar{e}_{ij}$. We shall have use of this result in §III of this paper.

Let R_2 denote the ring of all 2×2 matrices with coefficients in R. R_2 is naturally equipped with an involution $x \to \bar{x}$ satisfying $x + \bar{x} \in R$ and $x\bar{x} \in R$. Namely, if $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\bar{x} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We shall say that an alternative algebra Λ over R is a split Cayley algebra if $\Lambda \cong R_2 \oplus vR_2$ where v is an element such that $v^2 = 1$, and $xv = v\bar{x}$ for all $x \in R_2$.

Suppose Λ is an alternative algebra over a local ring R. Let m be the unique maximal ideal of R. If Λ is finitely generated over R, then the radical J of Λ is well defined. Since $J \supset m\Lambda$, Λ/J is naturally an R/m-algebra.

Theorem 3. *Let Λ be an alternative algebra over a Hensel ring R. Suppose Λ is finitely generated over R, and Λ/J is a split Cayley algebra over R/m. Then there exists a split Cayley algebra Λ_0 over R contained in Λ such that $\pi(\Lambda_0) = \Lambda/J$.*

Proof. In this proof, we denote $\pi(x)$ by $[x]$ for $x \in \Lambda$. By hypothesis, $\Lambda/J = (R/m)_2 \oplus [\omega](R/m)_2$ is a split Cayley algebra over the field R/m. Now by Theorem 2, there exist matrix units $\{e_{ij} \mid i, j = 1, 2\}$ in Λ such that $(R/m)_2 = \Sigma^2_{i,j=1} (R/m)[e_{ij}]$. Set $e = e_{11} + e_{22}$. Then $[e] = 1$ the identity element of Λ/J. Thus, $1 - e \in J$. By Proposition 1, e is a unit in Λ. So, there exists a $z \in \Lambda$ such that $ez = ze = 1$. Since $R[z, e]$ is an associative subalgebra of Λ, we have $0 = [(1-e)e]z = (1-e)(ez) = 1 - e$. Thus $e_{11} + e_{22} = 1$.

The proof from this point on is similar to [9, 3.21]. We shall borrow freely from this result. Find f_{ij} $(i \neq j) \in \Lambda$ such that $[f_{ij}] = [\omega][e_{jj}]$. We may assume $f_{ij} \in e_{ii}\Lambda e_{jj}$. Also $e_{ji} f_{ij} = c_j \in J \cap \Lambda_{jj}$. Here Λ_{11}, Λ_{12}, Λ_{21} and Λ_{22} are the Peirce spaces arising from the idempotents e_{11} and e_{22} in Λ. Set $h_{ij} = f_{ij} - e_{ij}c_j$ for $i \neq j$. Then $h_{ij} \in \Lambda_{ij}$, $[h_{ij}] = [f_{ij}]$ and $e_{ji}h_{ij} = h_{ij}e_{ji} = 0$ for $i \neq j$. We also have $[h_{ij}][h_{ji}] = [f_{ij}][f_{ji}] = [e_{ii}]$. Thus, $h_{ij}h_{ji} = e_{ii} - a_i$ for some element $a_i \in J \cap \Lambda_{ii}$. Now $\Lambda_{ii} = e_{ii}\Lambda e_{ii}$; and, by Proposition 2, $J \cap \Lambda_{ii}$ is a quasi-regular ideal in Λ_{ii}. Thus, there exists an element $a'_i \in \Lambda_{ii}$ such that $(e_{ii} - a_i)(e_{ii} - a'_i) = e_{ii} = (e_{ii} - a'_i)(e_{ii} - a_i)$. Furthermore, Proposition 3 implies that a'_i is a polynomial in a_i without constant term. Now set

$$p_{12} = (e_{11} - a'_1)h_{12} \quad \text{and} \quad p_{21} = h_{21}. \tag{8}$$

We note that $p_{12} \in \Lambda_{12}$, $p_{21} \in \Lambda_{21}$, $[p_{12}] = [h_{12}] = [f_{12}]$ and $[p_{21}] = [h_{21}] = [f_{21}]$. We wish to show that $p_{ij}p_{ji} = e_{ii}$ for $i \neq j$. Now $p_{12}p_{21} = [(e_{11} - a'_1)h_{12}]h_{21}$. These elements have the form $e_{11} - a'_1 = e_{11}\lambda_1 e_{11}$, $h_{12} = e_{11}\lambda_2 e_{22}$ and $h_{21} = e_{22}\lambda_3 e_{11}$ for some λ_1, λ_2 and $\lambda_3 \in \Lambda$. From this, one can easily argue that $(e_{11} - a'_1, h_{12}, h_{21}) = 0$. Thus, $p_{12}p_{21} = (e_{11} - a'_1)(h_{12}h_{21}) = (e_{11} - a'_1)(e_{11} - a_1) = e_{11}$. To show that $p_{21}p_{12} = e_{22}$, we need the following identities,

$$a_i h_{ij} = h_{ij} a_j, \tag{9}$$

$$(i \neq j).$$

$$a'_i h_{ij} = h_{ij} a'_j, \tag{10}$$

For (9), we have $a_i h_{ij} = (e_{ii} - h_{ij}h_{ji})h_{ij} = h_{ij} - (h_{ij}h_{ji})h_{ij} = h_{ij} - h_{ij}(h_{ji}h_{ij}) = h_{ij} - h_{ij}(e_{jj} - a_j) = h_{ij} - h_{ij} + h_{ij}a_j = h_{ij}a_j$. Therefore, (9) is proven.

(10) follows from (9) and the fact that a'_i is a polynomial in a_i. We first note that $(a_i, h_{ij}, a_j) = 0$. Thus using (9), we have

(11) $$(b_{ij}a_j)(e_{jj}-a_j)=(e_{ii}-a_i)(b_{ij}a_j).$$

Now $e_{ii}-a_i' \in Re_{ii}[a_i] \subset \Lambda_{ii}$. Thus, $e_{ii}-a_i'$ associates with $e_{ii}-a_i$ and $(b_{ij}a_j)$. Thus from (11), we get

(12) $$(e_{ii}-a_i')[(b_{ij}a_j)(e_{jj}-a_j)]=b_{ij}a_j.$$

One can now directly argue that $(e_{ii}-a_i', b_{ij}a_j, e_{jj}-a_j)=0$. Thus, multiplying both sides of (12) by $(e_{ii}-a_i')$, we get

(13) $$(e_{ii}-a_i')(b_{ij}a_j)=(b_{ij}a_j)(e_{jj}-a_j').$$

Since $a_i+a_i'-a_i'a_i=0$, we get $a_i'=(-a_i)(e_{ii}-a_i')$. Thus using (9), (13) and the fact that $(e_{ii}-a_i')$ is a polynomial in a_i with coefficients in Re_{ii}, we get

$$a_i'b_{ij}=[(-a_i)(e_{ii}-a_i')]b_{ij}=(e_{ii}-a_i')(-a_ib_{ij})=(e_{ii}-a_i')(-b_{ij}a_j)$$

$$=(-b_{ij}a_j)(e_{jj}-a_j')=b_{ij}[(-a_j)(e_{jj}-a_j')]=b_{ij}a_j'.$$

Therefore (10) is proven.

We can now prove $p_{21}p_{12}=e_{22}$. We first note that (9) and (10) imply $p_{12}=(e_{11}-a_1')b_{12}=b_{12}-a_1'b_{12}=b_{12}-b_{12}a_2'=b_{12}(e_{22}-a_2')$. Thus, $p_{21}p_{12}=b_{21}[b_{12}(e_{22}-a_2')]=(b_{21}b_{12})(e_{22}-a_2')=(e_{22}-a_2)(e_{22}-a_2')=e_{22}$.

We next note that $e_{ij}p_{ji}=p_{ji}e_{ij}=0$ if $i\neq j$. For $e_{12}p_{21}=e_{12}b_{21}=0$ and $e_{21}p_{12}=e_{21}\cdot(e_{11}-a_1')b_{12}=e_{21}b_{12}\cdot(e_{22}-a_2')=0$. Now set $v=p_{12}+p_{21}$. We note that $p_{ij}\in\Lambda_{ij}$, and thus $p_{ij}^2=0$ by [9, 3.20]. So $v^2=(p_{12}+p_{21})^2=p_{12}p_{21}+p_{21}p_{12}=e_{11}+e_{22}=1$. We also have that $[v]=[w]$. Set $R_2=\Sigma_{i,j=1}^2 Re_{ij}\subset\Lambda$. Then , $v\notin R_2$. One can easily show $vR_2\cap R_2=0$. Thus, we can consider $\Lambda_0=R_2\oplus vR_2\subset\Lambda$. Clearly $\pi(\Lambda_0)=\Lambda/J$. So the theorem will be complete if we show that $xv=v\bar{x}$ for all $x\in R_2$.

Let $x\in R_2$. Then $x=ae_{11}+be_{12}+ce_{21}+de_{22}$ for constants a, b, c and $d\in R$. Then $\bar{x}=de_{11}-be_{12}-ce_{21}+ae_{22}$, $xv=ap_{12}+be_{12}p_{12}+ce_{21}p_{21}+dp_{21}$, and $v\bar{x}=ap_{12}-bp_{12}e_{12}-cp_{21}e_{21}+dp_{21}$. But if $x_{ij}, y_{ij}\in\Lambda_{ij}$ $(i\neq j)$, then $x_{ij}y_{ij}=-y_{ij}x_{ij}$ by [9, 3.21]. Thus, $xv=v\bar{x}$. □

In Theorem 3, if we replace J by an ideal I whose square is zero, then we could drop the hypothesis that Λ is finitely generated over R. We would then have the following result: Suppose Λ is an alternative algebra over R, and I is an ideal in Λ such that $I^2=0$. Suppose Λ/I is a split Cayley algebra over $R/I\cap R$. Then there exists a split Cayley algebra Λ_0 over R contained in Λ such that $\pi(\Lambda_0)=\Lambda/I$. A proof of this remark would follow from the remark after Theorem 2 and the proof of [9, Lemma 3.21]. □

III. A Wedderburn theorem for alternative algebras over complete local rings of equal characteristic. Throughout this section, we assume R is a split local ring. Thus, $\bar{R}=R/m$ is imbedded in R via some ring monomorphism ϵ. We shall drop ϵ and consider $\bar{R}$ as contained in R.

Proposition 6. *Let Λ be an alternative algebra (not necessarily finitely generated) over a split local ring R. Suppose I is an ideal in Λ which contains $m\Lambda$ and has square zero. Then if Λ/I is central separable over $\overline{R}$, $0 \to I \to \Lambda \to \Lambda/I \to 0$ splits as $\overline{R}$-algebras.*

Proof. It is clearly sufficient to show that there exists an $\overline{R}$-subalgebra S of Λ such that $I \cap S = 0$ and $I + S = \Lambda$. Since Λ/I is central separable over the field $\overline{R}$, Λ/I is either a Cayley algebra of dimension eight over $\overline{R}$, or Λ/I is $n \times n$ matrices over an associative division algebra D. In the latter case, D has center $\overline{R}$ and is finite dimensional over $\overline{R}$. In either case, let K be a field which contains $\overline{R}$, is finite dimensional over $\overline{R}$ and splits Λ/I, i.e. $\Lambda/I \otimes_{\overline{R}} K$ is either a split Cayley algebra over K or $n' \times n'$ matrices $K_{n'}$ over K. Since Λ/I is finite dimensional over $\overline{R}$, such a field K exists. Write $K = \overline{R}(\epsilon_1, \cdots, \epsilon_r)$ with $\epsilon_1 = 1$.

Now consider $\Lambda \otimes_{\overline{R}} K$. This is an alternative algebra over the field K, and $I \otimes_{\overline{R}} K$ is an ideal nilpotent of index two in $\Lambda \otimes_{\overline{R}} K$. Furthermore, $(\Lambda \otimes_{\overline{R}} K)/(I \otimes_{\overline{R}} K) \cong (\Lambda/I) \otimes_{\overline{R}} K$. Thus, the remarks made after Theorems 2 and 3 when applied to $\Lambda \otimes_{\overline{R}} K$ imply that there exists a K-subalgebra S' of $\Lambda \otimes_{\overline{R}} K$ such that $S' \oplus (I \otimes_{\overline{R}} K) = \Lambda \otimes_{\overline{R}} K$. The proof now proceeds exactly as in the associative case [1, pp. 47–48] to obtain an $\overline{R}$-subalgebra S of Λ with $S \oplus I = \Lambda$. □

Proposition 7. *Suppose Λ is an alternative algebra (not necessarily finitely generated) over a split local ring R. Suppose I is an ideal in Λ which contains $m\Lambda$ and has square zero. Then if Λ/I is separable over $\overline{R}$, $0 \to I \to \Lambda \to \Lambda/I \to 0$ splits as $\overline{R}$-algebras.*

Proof. Λ/I separable over $\overline{R}$ means Λ/I is finite-dimensional over $\overline{R}$, and Λ/I decomposes into a finite direct sum of ideals $S_1 \oplus \cdots \oplus S_n$. Each S_i is central simple over its center Z_i which in turn is a separable field over $\overline{R}$. Let the identities of the subalgebras S_i be $\overline{e}_1, \cdots, \overline{e}_n$. Then we can lift these pairwise orthogonal idempotents to pairwise orthogonal idempotents $e_1, \cdots, e_n$ in Λ. Since $\overline{e}_1 + \cdots + \overline{e}_n = 1$, it follows that $e_1 + \cdots + e_n = 1$ in Λ. We also know that $0 \to e_i I e_i \to e_i \Lambda e_i \to S_i \to 0$ is exact. Thus, for the purposes of proving the proposition, we may assume Λ/I is central simple over its center Z, and that Z is a finite separable field extension of $\overline{R}$.

Hence, we may assume that Λ/I is either $n \times n$-matrices over an associative division algebra D having center Z, or that Λ/I is a Cayley algebra over Z. In either case, since Z is a finite separable extension of $\overline{R}$, there exists a field F such that F is a finite separable extension of $\overline{R}$, and $Z \otimes_{\overline{R}} F \cong \bigoplus_{i=1}^{l} \overline{e}_i F$ for pairwise orthogonal idempotents $\overline{e}_1, \cdots, \overline{e}_l$ in $Z \otimes_{\overline{R}} F$. Consider the exact sesequence of F-algebras: $0 \to I \otimes_{\overline{R}} F \to \Lambda \otimes_{\overline{R}} F \to \Lambda/I \otimes_{\overline{R}} F \to 0$. If Λ/I is associative, then the center of $\Lambda/I \otimes_{\overline{R}} F$ is $Z \otimes_{\overline{R}} F$ [2, Corollary 1.6]. If Λ/I is nonassociative, then Λ/I is a Cayley algebra over Z. Thus, $\Lambda/I = Q \oplus vQ$ for some quaternian algebra Q over Z. So $Q \otimes_{\overline{R}} F$ is an associative subalgebra of $\Lambda/I \otimes_{\overline{R}} F$. Hence the center $C(\Lambda/I \otimes_{\overline{R}} F)$ of $\Lambda/I \otimes_{\overline{R}} F$ is contained in the center $C(Q \otimes_{\overline{R}} F)$ of $Q \otimes_{\overline{R}} F$. Now Q is central separable over Z, and Z is separable over $\overline{R}$. Thus, Q is separable over $\overline{R}$. It again follows from [2, Corollary 1.6] that $C(Q \otimes_{\overline{R}} F) = Z \otimes_{\overline{R}} F$. Thus, in either case the center of $\Lambda/I \otimes_{\overline{R}} F$ is $Z \otimes_{\overline{R}} F = \bigoplus_{i=1}^{l} F\overline{e}_i$.

Now from the remark after Theorem 1, we may lift $\{\overline{e}_1, \cdots, \overline{e}_l\}$ to pairwise orthogonal idempotents $e_1, \cdots, e_l$ in $\Lambda \otimes_{\overline{R}} F$ whose sum is 1. For each $i = 1, \cdots, l$, $0 \to e_i(I \otimes_{\overline{R}} F)e_i \to e_i(\Lambda \otimes_{\overline{R}} F)e_i \to (\Lambda/I \otimes_{\overline{R}} F)\overline{e}_i \to 0$ is an exact sequence of F-algebras. $e_i(I \otimes_{\overline{R}} F)e_i$ is a nilpotent ideal of index two, and the center of $(\Lambda/I \otimes_{\overline{R}} F)\overline{e}_i$ is isomorphic to F. Since $\Lambda/I \otimes_{\overline{R}} F$ is separable over F, $(\Lambda/I \otimes_{\overline{R}} F)\overline{e}_i$ is central separable over F. Thus, Proposition 6 implies there exists an F-subalgebra S_i of $e_i(\Lambda \otimes_{\overline{R}} F)e_i$ such that $S_i \oplus e_i(I \otimes_{\overline{R}} F)e_i = e_i(\Lambda \otimes_{\overline{R}} F)e_i$. Setting $S' = S_1 + \cdots + S_l$, we get an F-subalgebra of $\Lambda \otimes_{\overline{R}} F$ such that $(I \otimes_{\overline{R}} F) \oplus S' = \Lambda \otimes_{\overline{R}} F$. Following the proof of Wedderburn's theorem in [1, pp. 47–48], we can then find an $\overline{R}$-subalgebra S of Λ such that $S \oplus I = \Lambda$. This immediately implies $0 \to I \to \Lambda \to \Lambda/I \to 0$ splits. □

We can now prove the main theorem.

Theorem 4. *Let Λ be an alternative algebra over a complete local ring R of equal characteristic. Suppose Λ is finitely generated as an R-module, and Λ/J is separable over $\overline{R} = R/m$. Then there exists an $\overline{R}$-subalgebra S of Λ such that $S + J = \Lambda$ and $S \cap J = 0$.*

Proof. We had noted in the preliminaries that a complete local ring R of equal characteristic contains a copy of its residue class field $\overline{R}$. We choose a copy of $\overline{R}$ in R and regard Λ as an $\overline{R}$-algebra via this copy.

For each $n \geq 1$, define $J^{2^n} = J^{2^{n-1}} J^{2^{n-1}}$. We first prove that Λ is a complete Hausdorff space in its J^{2^n}-adic topology. By definition, J is the complete inverse image of the radical of $\Lambda/m\Lambda$. Since Λ is finitely generated as an R-module, $\Lambda/m\Lambda$ is a finite-dimensional alternative algebra. Hence, the radical of $\Lambda/m\Lambda$ is nilpotent. Therefore, there exists an $n_0 > 0$ such that $J^{2^{n_0}} \subset m\Lambda$. This implies $\bigcap_{n=1}^{\infty} J^{2^n} \subset \bigcap_{n=1}^{\infty} m^n \Lambda$. But by [11, Theorem 9, p. 262], $\bigcap m^n \Lambda = 0$. Thus, Λ is a Hausdorff space in its J^{2^n}-adic topology. By [11, Theorem 5, p. 256], Λ is a complete Hausdorff space in its $m\Lambda$-adic topology. Thus, Λ is complete in its J^{2^n}-adic topology.

Now for each $n \geq 1$, we have

(14) $$0 \to J/J^{2^n} \to \Lambda/J^{2^n} \to \Lambda/J \to 0$$

is an exact sequence of alternative algebras. Since Λ/J is separable over $\bar{R}$, one can easily argue, using Proposition 4, that J/J^{2^n} is the radical of Λ/J^{2^n}. If $n = 1$, Proposition 7 implies (14) splits as $\bar{R}$-algebras. Hence, there exists an $\bar{R}$-subalgebra S_2 of Λ/J^2 such that $S_2 \oplus J/J^2 = \Lambda/J^2$. In particular, S_2 is isomorphic to Λ/J. If $n = 2$, we have the following commutative diagram with exact rows:

(15) $$\begin{array}{ccccccccc} 0 & \to & J/J^2 & \to & \Lambda/J^2 & \xrightarrow{\pi_2} & \Lambda/J & \to & 0 \\ & & \uparrow x_2 & & \uparrow x_2 & & \| & & \\ 0 & \to & J/J^4 & \to & \Lambda/J^4 & \xrightarrow{\pi_4} & \Lambda/J & \to & 0 \end{array}$$

Here x_2 is the natural projection of Λ/J^4 onto Λ/J^2. Since x_2 is an R-algebra homomorphism, x_2 is also an $\bar{R}$-algebra homomorphism. Set $S_4' = x_2^{-1}(S_2)$. Then S_4' is an $\bar{R}$-subalgebra of Λ/J^4. We note that

(16) $$0 \to J^2/J^4 \to S_4' \to S_2 \cong \Lambda/J \to 0$$

is an exact sequence of $\bar{R}$-algebras. Thus, by Proposition 7, (16) splits. Hence, there exists an $\bar{R}$-subalgebra S_4 of Λ/J^4 such that $S_4 \oplus J^2/J^4 = S_4'$. Since $x_2(S_4) = S_2$, we have $\pi_4(S_4) = \Lambda/J$. From this it follows that $S_4 \oplus J/J^4 = \Lambda/J^4$. Thus (14) splits when $n = 2$.

By induction, we can show that for each $n \geq 1$ there exists an $\bar{R}$-subalgebra S_{2n} of Λ/J^{2^n} such that $S_{2n} \oplus J/J^{2^n} = \Lambda/J^{2^n}$ and $x_{2n+1}(S_{2n+1}) = S_{2n}$. Since Λ is complete in its J^{2^n}-adic topology, the inverse limit $\varprojlim \Lambda/J^{2^n}$ is isomorphic to Λ. If we set $S = \varprojlim S_{2n}$ (see [11, pp. 305–306]), then S is an $\bar{R}$-subalgebra of Λ for which $S \oplus J = \Lambda$. □

IV. **A conjecture concerning Hensel rings.** One may ask if there is a broader class of rings for which Theorem 4 holds. The completeness of R was used heavily in the proof of Theorem 4, but Theorems 1, 2 and 3 seem to suggest that Theorem 4 may hold if R is a split Hensel ring. The author conjectures that Theorem 4 holds if we assume R is a split Hensel ring instead of R being a complete local ring. In view of [5, Theorem], this would be the broadest class of local rings for which one could hope to prove Theorem 4.

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